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H_2 Approach for Optimally Tuning Passive Vibration Absorbers to Flexible Structures

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Introduction

PASSIVE damping by vibration absorbers has been proven both reliable and successful in many areas of vibration control. Research into the optimal tuning of vibration absorbers has received increased attention lately; however, these tuning methods have only been practical on low-order systems (one or two modes).[1-4]

The design of a vibration absorber system is an optimization problem. In the general case, it is desired to couple multiple vibration absorbers to a structure with multiple vibrational modes. If the total mass of all of the actuators is fixed to some small value compared to the mass of the structure, then the design parameters are the set of absorber stiffnesses and damping coefficients. The problem then becomes the optimization of these parameters with respect to some design criteria. Den Hartog[5] attempted to minimize the maximum value of the transfer function. Juang[2] attempted to minimize a quadratic cost. Both of these approaches give similar tuning values. However, they are only guaranteed to minimize the design criteria for the case of a single vibration absorber tuned to a single vibrational mode. Minimizing the design criteria for the multiple mode and absorber case results is a nonlinear optimization problem. Miller and Crawley[1] experimentally investigated passive absorbers. They validated some general guidelines for tuning vibration absorbers to multiple modes.

There is no straightforward design approach for tuning multiple vibration absorbers to a flexible structure. Also, in some circumstances it may be desirable to design vibration absorbers in the time domain. In this Note it is shown that by using a standard quadratic performance index, output feedback regulator theory provides an approach for tuning vibration absorbers to flexible structures. The technique is computationally attractive, and since it is based on linear quadratic regulator (LQR) design techniques, provides a time domain design method for tuning vibration absorber systems.

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Passive Absorber Tuning by Output Feedback

The following feedback formulation of the passive absorber problem was developed by Posbergh et al.[3]:

$$\dot{x}(t) = Ax(t) + Bf(t) \tag{1}$$

where A and B are the augmented matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_s}{m_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \dfrac{1}{m_s} \\ 0 \\ -\dfrac{1}{m_a} \end{bmatrix}$$

where m_s is the mass of the structure, m_a the mass of the actuator, k_s the stiffness of the structure, and $x(t) = [x_s(t)\ \dot{x}_s(t)\ x_a(t)\ \dot{x}_a(t)]^T$. The structure-actuator system is coupled by the output feedback,

$$f(t) = k_a[x_s(t) - x_a(t)] + c_a[\dot{x}_s(t) - \dot{x}_a(t)] \tag{2}$$

which can be put in the matrix form,

$$f(t) = KCx(t) \tag{3}$$

where k_a and c_a are the stiffness and damping of the actuator, and K is a matrix of these values. The matrix C generates the relative position and velocity measurements. Notice that although the feedback formulation was developed based on the single-degree-of-freedom (SDOF) case, it is equally valid for the multiple degree of freedom case.

In this formulation of the passive absorber problem it is interesting to note that tuning the vibration absorber amounts to finding the gains that force the system to meet some design criteria, it is a feedback control problem. A variety of design approaches can be used to solve the problem. The following approach is based on LQR theory.

Given the output feedback problem represented by Eqs. (1) and (3), the passive absorber tuning problem is to find the gain matrix K that minimizes the performance index

$$J = \int_0^\infty x^T(t)Qx(t)\,\mathrm{d}t \tag{4}$$

Because full state feedback is not available, there are natural bounds restricting where the poles can be assigned, and consequently R is set to zero. Also, because passive vibration absorbers are typically performance limited, it is assumed that the desire is to design an absorber that will operate at maximum performance.

Levine and Athans[6] and Kosut[7] investigated the general output feedback problem where $R > 0$. In the specific case where R is zero, it can be shown that the H_2 cost simplifies to

$$J = \mathrm{tr}(P) \tag{5}$$

where P satisfies

$$(A - BCK)^T P + P(A - BKC) + Q = 0 \tag{6}$$

and the necessary conditions for this cost function are

$$B^T VPC^T = 0 \tag{7}$$

where V satisfies

$$(A - BKC)^T V + V(A - BKC) + Q = 0 \tag{8}$$

and P satisfies

$$P(A - BKC)^T + (A - BKC)P + I = 0 \tag{9}$$

Direct use of the necessary conditions to minimize the cost function requires the solution of a nonlinear matrix equation. Because the algebraic cost function (5) and (6) is simple to evaluate, a nonlinear optimization using the algebraic cost function would seem more attractive. There are, however, a number of problems that need to be addressed to perform this optimization, for example, determination of the starting conditions and likelihood of convergence. If the starting conditions can be chosen very close to the optimal values, then convergence is much more likely and computational effort may be small. Medanic and Uskokovic[8] formulated an alternate approach to solving the optimal output regulator problem. This approach is useful for directly solving the H_2 tuning problem or as a first step for determining the starting conditions for an H_2 optimization.

Output Regulator Design by Projection

Medanic and Uskokovic[8] formulated a computationally attractive approach to solving the optimal output feedback problem. Briefly, for the closed-loop system,

$$\dot{x} = (A - BK)x = A_{c_l}x \tag{10}$$

the full state regulator that minimizes the quadratic performance index,

$$J = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)\,dt \tag{11}$$

is given by

$$u = -R^{-1}B^TMx \tag{12}$$

and M is the solution of

$$A^TM + MA - MBR^{-1}B^TM + Q = 0 \tag{13}$$

Let X and Λ denote the eigenvectors and eigenvalues of A_{cl}. Let $X_r \in R^{n \times r}$ be a matrix formed from r generalized eigenvectors of A_{cl} associated with the eigenvalues Λ_r. Now define the projection matrix

$$P = X_r(CX_r)^{-1}C \tag{14}$$

then the subspace control will be

$$u = -R^{-1}B^TPx \tag{15}$$

The idea behind the method of Ref. 8 is to find an invariant subspace of the full state regulator that meets the dimension requirement imposed by the output feedback. The projection defined by Eq. (14) retains an invariant subspace of the full state feedback gains that minimizes the performance index. The choice of eigenvectors for the projection operator determines the modes of the system into which the control energy will be applied.

The method of Medanic and Uskokovic[8] is computationally attractive. The method requires the solution to the steady state Riccati equation and a few matrix multiplications. Using MATLAB-type computer software it is easily applied, even to very large systems.

Experimental Results

Using optimal output regulator theory a single vibration absorber was tuned to a low-order experimental structure, nicknamed the mass reactive T (MRT). It is shown in Fig. 1. This structure was designed to possess two isolated low-frequency vibrational modes. The two low-frequency modes of interest consisted of a torsional mode and a bending mode.

The frequency of the torsional mode could be changed by moving the attached weights, whereas the frequency of the bending mode remained fixed at 5.075 Hz. Thus, by moving the weights different modal spacing could be obtained. An actuator is located at one end of the structure. The absorber-to-structure mass ratio was approximately 0.002. Using a two-mode system-identified model of the structure, the subspace assignment method was used to tune a single vibration absorber to the first mode of the structure, then the second mode, and finally to both modes. The choice of eigenvectors used in the projection matrix determines the modal tuning. An H_2 optimization was then performed to determine the H_2 optimal tuning parameters. The subspace assignment and optimal tuning were performed on three different modal configurations, 0.5-, 2-, and 5-Hz modal separations.

Tables 1–3 give the actuator frequency and damping obtained by computer simulation of the subspace assignment method and the H_2 optimization. They also give the cost obtained using those frequencies and damping to tune the actuator. For all of

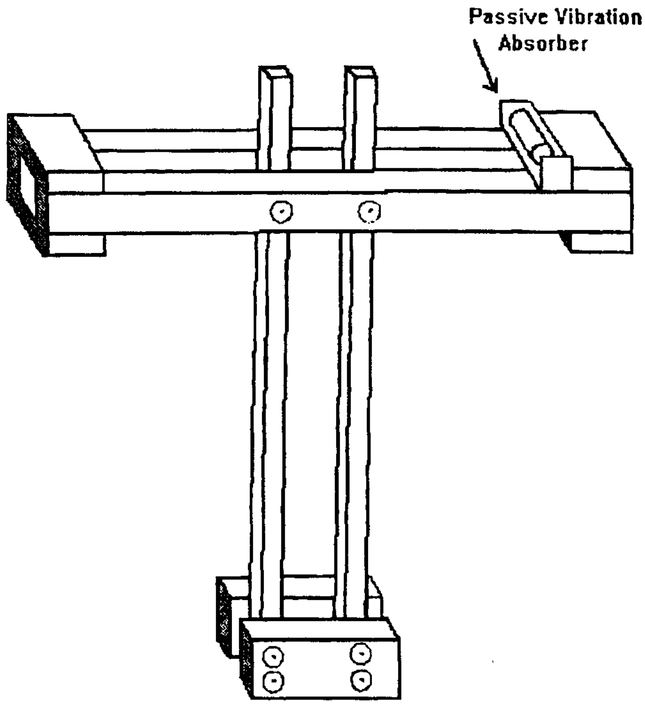

Fig. 1 Mass reactive T, experimental structure.

Table 1 Tuning parameters for 2-Hz modal separation

Modes	Closed-loop H_2 cost	Actuator frequency, Hz	Actuator damping
First	0.5434	5.18	0.168
Second	0.9905	6.42	0.226
Both	0.6498	5.66	0.235
H_2 optimal	0.5252	5.34	0.137

Table 2 Tuning parameters for 0.5-Hz modal separation

Modes	Closed-loop H_2 cost	Actuator frequency, Hz	Actuator damping
First	0.8950	4.54	0.172
Second	0.8331	4.68	0.149
Both	0.8604	4.61	0.161
H_2 optimal	0.7770	4.71	0.1022

Table 3 Tuning parameters for 5-Hz modal separation

Modes	Closed-loop H_2 cost	Actuator frequency, Hz	Actuator damping
First	1.2739	5.18	0.153
Second	5.9310	9.63	0.211
Both	1.5103	7.04	0.3688
H_2-optimal	1.0069	5.61	0.2609

the vibration absorber designs only the position of the structure was weighted. Because the goal was to minimize the displacement of the structure, all other entries in the Q matrix are set to zero.

Results indicate that in all cases, the subspace assignment method alone gives nearly optimal results. The resulting values for the vibration absorber for the 0.5-Hz and 2.0-Hz modal separation were tested on an actual structure. The difference between the subspace assignment values and the H_2 optimal values were found to be within the tuning tolerance of the absorber.

Summary

This Note describes an approach for tuning vibration absorbers based on output regulator theory developed in Refs. 6–8. Because the method is based on well-developed LQR design techniques, the approach is efficient and computationally attractive. Two approaches are presented, design by subspace assignment and design by subspace assignment combined with numerical optimization. The two approaches are demonstrated on a structure with two dominant adjustable low-frequency modes. Results show the subspace assignment method to be effective at tuning one vibration absorber to more than one vibrational mode. The computationally attractive subspace assignment will result in a nearly optimal system.

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